2. Sets and Set Operations

Section 2.1: Sets

- **Set** is a collection of distinct unordered objects.
- Members of a set are called **elements.**
- For example:
  - $A = \{1, 3, 5, 7\}$; $3 \in A$, $2 \notin A$
  - $B = \{\text{Cat, Dog, Fox}\}$
  - $C = \{x \mid x = n^2 + 1, n \text{ is an integer}, 0 \leq n \leq 10\}$
  - $D = \{x \mid 1 \leq x \leq 4\} = [1, 4]$

Basic Notations for Sets

(1) **listing all of its elements** in curly braces:

- **Finite sets:**

$$V = \{a, e, i, o, u\}$$

$$S = \{1, 2, 3, \ldots, 99\}$$

- **Infinite sets:**

N = {0, 1, 2, …}, The set of natural numbers

Z = {…, -2, -1, 0, 1, 2, …}, The set of integers

Z^+ = {1, 2, 3, …}, The set of positive integers

R = The set of real numbers

Basic Notations for Sets

(2) **Set builder notation**: For any proposition $P(x)$ over any universe of discourse, $\{x \mid P(x)\}$ is the set of all x such that $P(x)$.

Example

"The set of all odd positive integers less than 10"

$O = \{x \mid x \text{ is an odd positive integer and } x < 10\}$

Examples

Use the set builder notation to describe the sets

$$A = \{1, 4, 9, 16, 25, \ldots\}$$

$$B = \{\text{a, d, e, h, m, o}\}$$

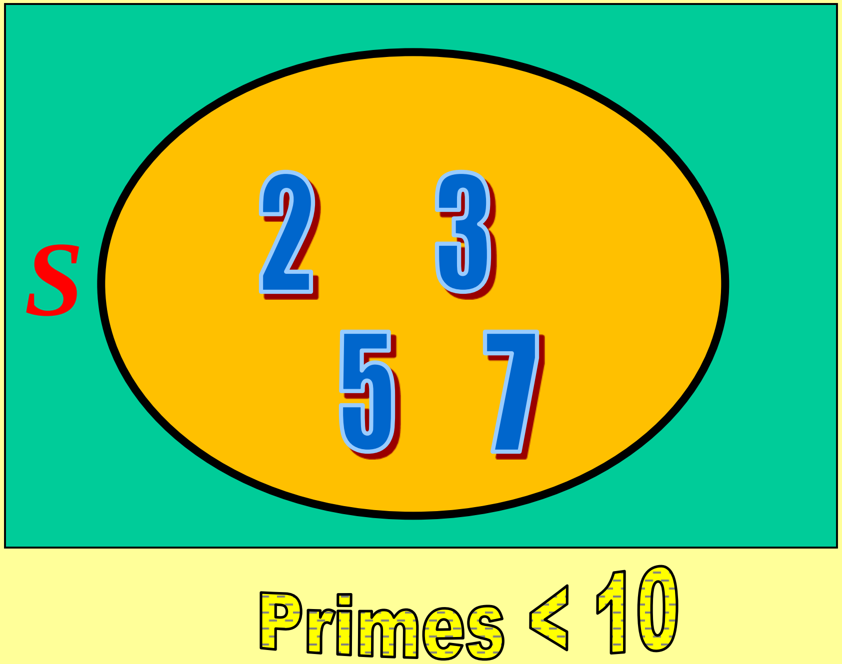
$$A = \{\, x \mid x = n^2 \text{ , } n \text{ is positive integer}\}$$

$$B = \{\, x \mid x \text{ is a letter of the word mohamed}\}$$

Basic Notations for Sets

(3) Venn Diagrams

U : universal set



Membership in Sets

- $x \in S$: x is an **element** or **member** of the set S.

  e.g. $3 \in \mathbf{N}$

  "a" $\in \{x \mid x$ is a letter of the alphabet$\}$
- $x \notin S :\equiv \neg(x \in S)$ "x is not in S"

The Empty Set

- $\varnothing$ (**null** or **the empty set**) is the unique set that contains no elements.
- $\varnothing = \{\}$
- **Empty set $\varnothing$** does not equal the **singleton set $\{\varnothing\}$**

$$\varnothing \neq \{\varnothing\}$$

Definition of Set Equality

- Two sets are declared to be **equal** if and only if they contain exactly the same elements.

i.e. A and B are equal if and only if

$$\forall x \,(x \in A \leftrightarrow x \in B)$$

Example: (Order and repetition do not matter)

{1, 3 , 5} = {3, 5, 1} = {1, 3, 3, 3, 5, 5, 5}

Subset Relation

- $S \subseteq T$ ("S is a **subset** of T") means that every element of S is also an element of T.
- $S \subseteq T \Leftrightarrow \forall x\, (x \in S \rightarrow x \in T)$
- $\varnothing \subseteq S$
- $S \subseteq S$
- If $S \subseteq T$ is true and $T \subseteq S$ is true then $S = T$,

$$\forall x\ (x \in S \leftrightarrow x \in T)$$

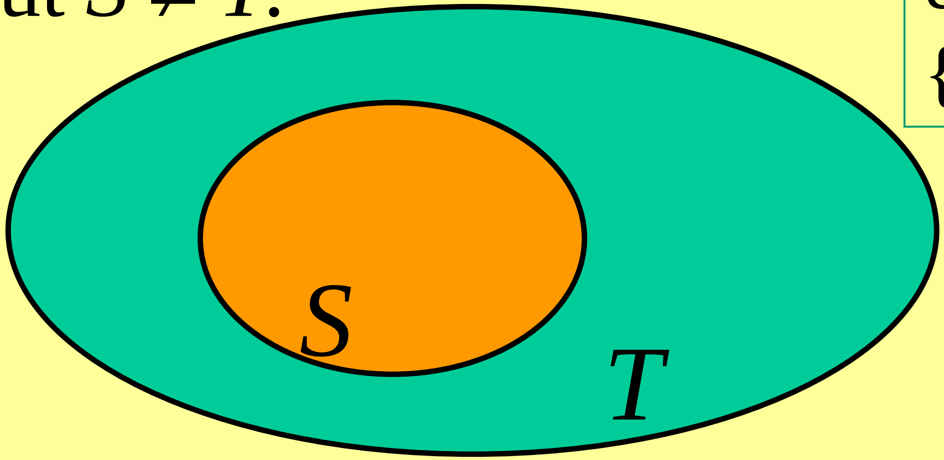
- $S \not\subseteq T$ means $\neg(S \subseteq T)$,

i.e. $\exists x\ (x \in S \wedge x \notin T)$

Proper Subset

- $S \subset T$ (S is a **proper subset** of T) means every element of S is also an element of T, but $S \neq T$.

e.g.
$\{1, 2\} \subset \{1, 2, 3\}$



Venn Diagram equivalent of $S \subset T$

Sets Are Objects, Too!

- The objects that are elements of a set may **themselves** be sets.

e.g. let $S = \{x \mid x \subseteq \{1,2,3\}\}$

then $S = \{\varnothing,$
$\{1\}, \{2\}, \{3\},$
$\{1,2\}, \{1,3\}, \{2,3\},$
$\{1,2,3\}\}$

- Note that $1 \neq \{1\} \neq \{\{1\}\}$!!!!

Cardinality and Finiteness

- $|S|$ or card(S) (read the **cardinality** of S) is a measure of how many different elements S has.

e.g. $|\varnothing| = |\{\ \}| = 0$

$|\{1, 2, 3, 5\}| = 4$

$|\{a, b, c\}| = 3$

$|\{\{1, 2, 3\},\{4, 5\}\}| = 2$

The Power Set Operation

- The **power** set $P(S)$ of a set S is the set of all subsets of S.

e.g.

- $P(\{a, b\}) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$
- $P(\{1, 2, 3\}) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\},$
  $\{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$
- $P(\{a\}) = \{\varnothing, \{a\}\}$
- $P(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}$
- $P(\varnothing) = \{\varnothing\}$

Note:

If a set has n elements then $P(S)$ has 2^n elements.

Ordered *n*-tuples

- The **ordered *n*-tuple** $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element and so on.
- These are like sets, except that duplicates matter, and the order makes a difference.

e.g. (2, 5, 6, 7) is a 4-tuple.

- **Note that $(1, 2) \neq (2, 1) \neq (2, 1, 1)$.**
- **Note: 2-tuples are called ordered pairs.**

Cartesian Products of Sets

- The **Cartesian product** of any two sets A and B is defined by

$$A \times B :\equiv \{(a, b) \mid a \in A \wedge b \in B\}.$$

e.g. $\{a, b\} \times \{1, 2\} = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$

- Note that for finite A and B, $|A \times B| = |A||B|$.
- Note that the Cartesian product is not commutative: $A \times B \neq B \times A$.

e.g. $\{1, 2\} \times \{a, b\} = \{(1, a), (1, b), (2, a), (2, b)\}$

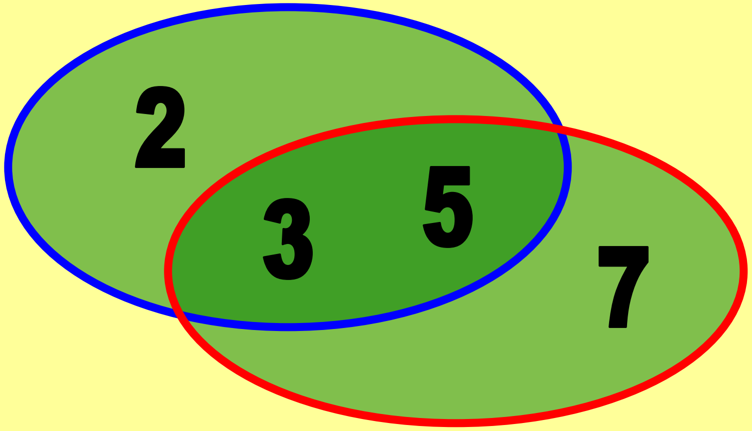
Section 2.2: Set Operations
The Union Operator $\cup$

- For any two sets A and B, $A \cup B$ is the set containing all elements that are either in A, or in B or in both.
- Formally:

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

Union Example

$\{2,3,5\} \cup \{3,5,7\} = \{2,3,5,3,5,7\} = \{2,3,5,7\}$



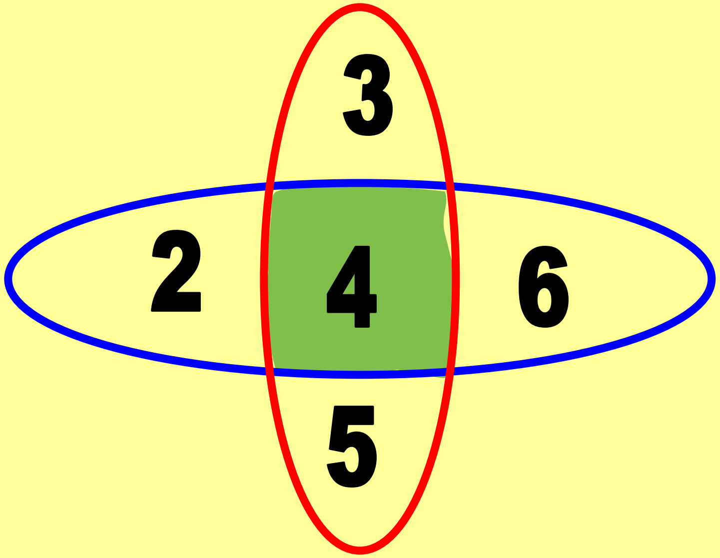
The Intersection Operator $\cap$

- For any two sets A and B, their **intersection** $A \cap B$ is the set containing all elements that are in both A and in B.
- Formally:

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$

Intersection Examples

- $\{a, b, c\} \cap \{2, 3\} = \varnothing$ **disjoint**
- $\{2, 4, 6\} \cap \{3, 4, 5\} = \{4\}$



Inclusion-Exclusion Principle

- How many elements are in $A \cup B$?

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example:

$\{1, 2, 3\} \cup \{2, 3, 4, 5\} = \{1, 2, 3, 4, 5\}$

$\{1, 2, 3\} \cap \{2, 3, 4, 5\} = \{2, 3\}$

$|\{1, 2, 3, 4, 5\}| = 3 + 4 - 2 = 5$

Set Difference

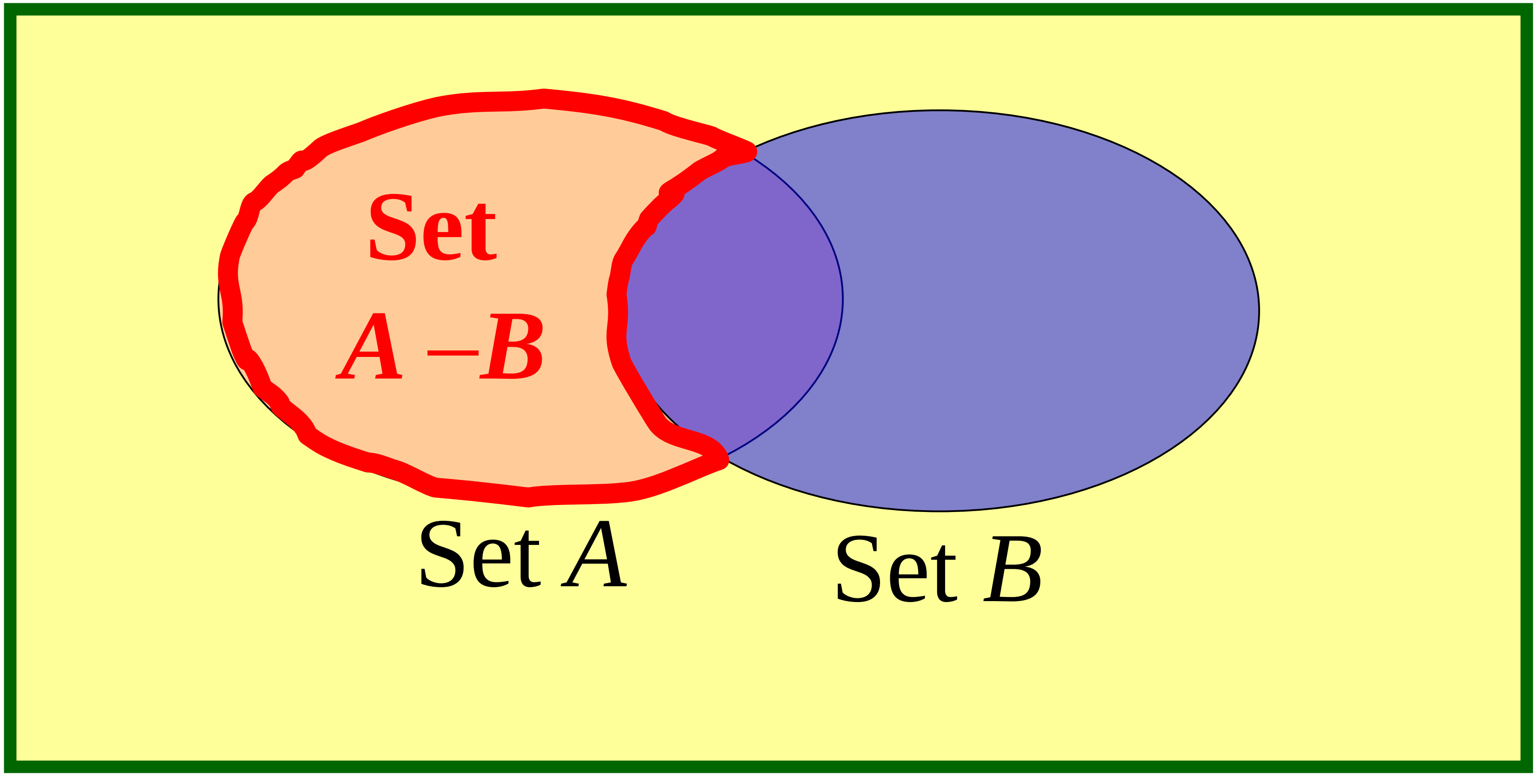
- For any two sets A and B, the **difference** of A and B, written $A - B$, is the set of all elements that are in A but not in B.

$$A - B :\equiv \{x \mid x \in A \wedge x \notin B\}$$
$$= \{x \mid \neg(x \in A \rightarrow x \in B)\}$$

$A - B = A \cap \overline{B}$ is called the **complement** of B **with respect to** A.

e.g. {1, 2, 3, 4, 5, 6} – {2, 3, 5, 7, 9, 11} = {1, 4, 6 }

Set Difference - Venn Diagram



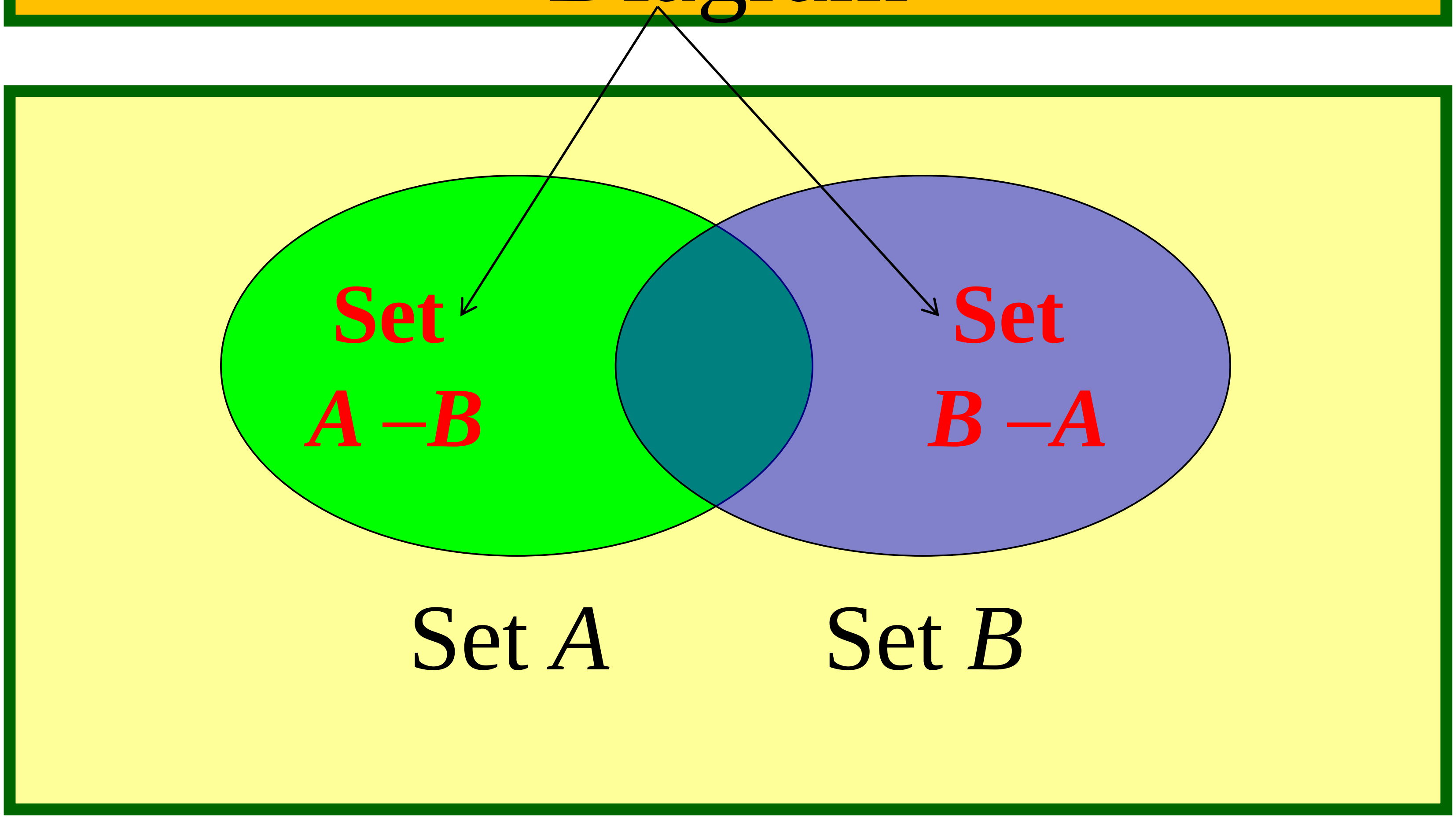
Symmetric Difference

- For any two sets A and B, the **symmetric difference** of A and B, written $A \oplus B$, is the set of all elements that are in A but not in B or in B but not in A.

$$A \oplus B :\equiv \{x \mid (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$$
$$= (A - B) \cup (B - A)$$
$$= (A \cup B) - (A \cap B)$$

e.g. $\{1, 2, 3, 4, 5, 6\} \oplus \{2, 3, 5, 7, 9, 11\} = \{1, 4, 6, 9, 11\}$

Symmetric Difference - Venn Diagram

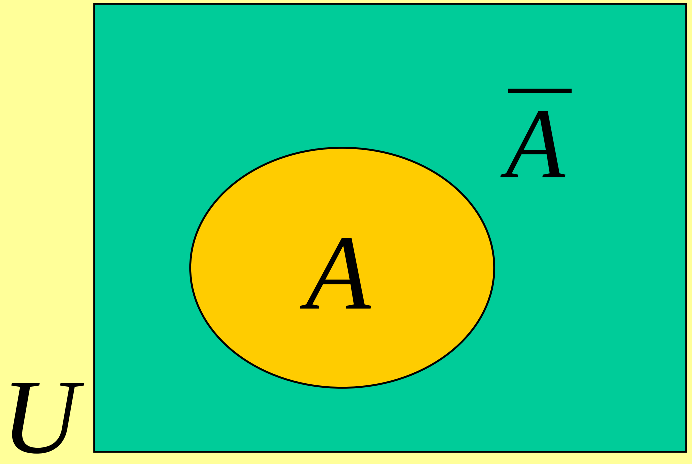


Set Complements

- U : Universe of Discourse

  $\overline{A}$: For any set $A \subseteq U$, the **complement** of A, i.e. it is $U - A$.

$$\overline{A} = \{x \mid x \notin A\}$$



e.g. If $U = \mathbf{N}$,

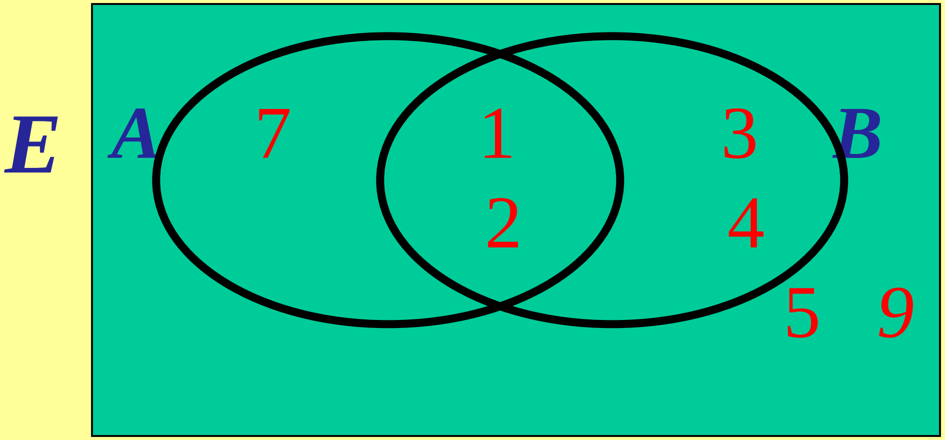
$$\overline{\{3,5\}} = \{0,1,2,4,6,7,\ldots\}$$

Example

Let A and B are two subsets of a set E such that $A \cap B = \{1, 2\}$, $|A| = 3$, $|B| = 4$, $\overline{A} = \{3, 4, 5, 9\}$ and $\overline{B} = \{5, 7, 9\}$. Find the sets A, B and E.

Example

Let A and B are two subsets of a set E such that $A \cap B = \{1, 2\}$, $|A| = 3$, $|B| = 4$, $\overline{A} = \{3, 4, 5, 9\}$ and $\overline{B} = \{5, 7, 9\}$. Find the sets A, B and E.



$A = \{1, 2, 7\}$, $B = \{1, 2, 3, 4\}$,

$E = \{1, 2, 3, 4, 5, 7, 9\}$

Set Identities

- Identity: $A \cup \varnothing = A = A \cap U$
- Domination: $A \cup U = U$, $A \cap \varnothing = \varnothing$
- Idempotent: $A \cup A = A = A \cap A$
- Double complement: $\overline{(\overline{A})} = A$
- Commutative: $A \cup B = B \cup A$, $A \cap B = B \cap A$
- Associative: $A \cup (B \cup C) = (A \cup B) \cup C$
  $A \cap (B \cap C) = (A \cap B) \cap C$
- Distribution: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- DeMorgan’s Law: $\overline{A \cup B} = \overline{A} \cap \overline{B}$
  $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Proving Set Identities

- To prove statements about sets of the form $E_1 = E_2$, where the Es are set expressions, there are three useful techniques:

1. Proving $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$ separately.
2. Using set builder notation and logical equivalences.
3. Using set identities.

Example: Show $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Method 1: Prove $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$

Assume $x \in \overline{A \cap B}$

$\Leftrightarrow x \notin A \cap B$ by the definition of the complement

$\Leftrightarrow \neg((x \in A) \wedge (x \in B))$ by the definition of intersection

$\Leftrightarrow \neg(x \in A) \vee \neg(x \in B)$ by De Morgan's law

$\Leftrightarrow x \notin A \vee x \notin B$ by the definition of negation

$\Leftrightarrow x \in \overline{A} \vee x \in \overline{B}$ by the definition of the complement

$\Leftrightarrow x \in \overline{A} \cup \overline{B}$ by the definition of union

Method 2: Set Builder Notation

Show $\overline{A \cap B} = \overline{A} \cup \overline{B}$

$$\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} \\
&= \{x \mid \neg\, (x \in (A \cap B))\} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} \\
&= \{x \mid \neg x \in A \vee \neg\, x \in B)\} \\
&= \{x \mid x \notin A \vee x \notin B\} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B})\} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} = \overline{A} \cup \overline{B}
\end{aligned}$$

Method 3 : Using Set Identities

Show that $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$

$$
\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cap (\overline{B \cap C}) && \text{De Morgan's law} \\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{De Morgan's law} \\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{Commutative law} \\
&= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{Commutative law}
\end{aligned}
$$

Method 3 : Using Set Identities

Show that $(B \cup C) - A = (B - A) \cup (C - A)$.

$$
\begin{aligned}
(B \cup C) - A &= (B \cup C) \cap \overline{A} \\
&= (B \cap \overline{A}) \cup (C \cap \overline{A}) \\
&= (B - A) \cup (C - A).
\end{aligned}
$$

Computer Representation of Sets

- Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. The **bit string** (of length $|U| = 10$) that represents the set $A = \{1, 3, 5, 6, 9\}$ has a one in the first, third, fifth, sixth, and ninth position, and zero elsewhere. It is

  1 0 1 0 1 1 0 0 1 0.